ON THE NOTION OF RELATIVE PROPERTY (T) FOR INCLUSIONS OF VON NEUMANN ALGEBRAS

by

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Abstract. We prove that the notion of *rigidity* (or *relative property* (T)) for inclusions of finite von Neumann algebras defined in [Po1] is equivalent to a weaker property, in which no "continuity constants" are required. The proof is by contradiction and uses infinite products of completely positive maps, regarded as correspondences.

The notion of *relative property* (T) (or *rigidity*) for inclusions of finite von Neumann algebras with countable decomposable center was introduced in ([P1]) by requiring that one of the following conditions (shown equivalent in [P1]) holds true:

(0.1). There exists a normal faithful tracial state τ on N such that: $\forall \varepsilon > 0$, $\exists F' = F'(\varepsilon) \subset N$ finite and $\delta' = \delta'(\varepsilon) > 0$ such that if $\mathcal{H}$ is a Hilbert N-bimodule with a vector $\xi \in \mathcal{H}$ satisfying the conditions $\|\langle \cdot \xi, \xi\rangle - \tau\| \leq \delta', \|\langle \xi \cdot, \xi\rangle - \tau\| \leq \delta'$ and $\|y\xi - \xi y\| \leq \delta', \forall y \in F'$, then $\exists \xi_0 \in \mathcal{H}$ such that $\|\xi_0 - \xi\| \leq \varepsilon$ and $b\xi_0 = \xi_0 b, \forall b \in B$.

(0.2). There exists a normal faithful tracial state τ on N such that: $\forall \varepsilon > 0$, $\exists F = F(\varepsilon) \subset N$ finite and $\delta = \delta(\varepsilon) > 0$ such that if $\phi : N \to N$ is a normal, completely positive (abreviated c.p. in the sequel) map with $\tau \circ \phi \leq \tau, \phi(1) \leq 1$ and $\|\phi(x) - x\|_2 \leq \delta, \forall x \in F$, then $\|\phi(b) - b\|_2 \leq \varepsilon, \forall b \in B$, $\|b\| \leq 1$.

(0.3). Condition (0.1) above is satisfied for any normal faithful tracial state τ on N.

(0.4). Condition (0.2) above is satisfied for any normal faithful tracial state τ on N.

This definition is the operator algebra analogue of the Kazhdan-Margulis relative property (T) for inclusions of groups $H \subset G$ ([M]). It is formulated in the same spirit Connes and Jones defined the property (T) for single von Neumann algebras in ([CJ]), starting from Kazhdan's property (T) for groups, by using Hilbert bimodules/c.p. maps, i.e., Connes' *correspondences* ([C2]). Thus, while in the case $H = G$ the relative property (T) of $H \subset G$ amounts to the property (T) of G, in the case $B = N$ and N is a factor the relative property (T) of $B \subset N$ in the sense of ([P1]) is equivalent to the property (T) of N in the sense of ([CJ]).

But there are in fact two possible ways to define the relative property (T) for inclusions of groups $H \subset G$: one requiring that all representations of G that have an almost

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G-invariant vector must contain a H-invariant vector, and another one with "continuity constants" of rigidity, requiring in addition that the vector fixed by H be close to the almost G-invariant vector. The first definition is the original one, formulated in ([M]). The second definition is obtained by adapting to the case of inclusions of groups a characterization of Kazhdan's property (T) "with continuity constants" which is implicit in ([DeKi], page 8), ([AW], Lemma 2) and appears explicitely in ([CJ], Proposition 1) or ([dHV], Proposition 1.16), a characterization that can be formulated both in terms of unitary representations and positive definite functions.

Conditions $(0.1) - (0.4)$ are all analogue to this second definition from group theory. The reason for opting for a definition "with continuity constants" for von Neumann algebras in ([P1]) is precisely its suitability to a formulation in terms of completely positive maps (the operator algebra substitute for positive definite functions), as well as its good behavior to tensor products and induction/reduction by projections.

For normal subgroups $H \subset G$ the two definitions of relative property (T) (with and without continuity constants) are easily seen to be equivalent: the same proof as in the single group case in (e.g., as in 1.16 of [dHV]) works. But for arbitrary $H \subset G$, this equivalence is non-trivial and was proved only recently by Jolissaint ([Jo]). While for applications it is important to have both definitions available as equivalent conditions, note that all known examples of subgroups $H \subset G$ with the relative property (T) are in fact normal (more precisely, H already has the relative property (T) in its normalizer in G).

Such equivalence is even more difficult to establish in the context of von Neumann algebras, where already the single von Neumann algebra case requires a delicate argument (cf. [CJ]). Yet it is desirable to have both types of characterizations. Thus, although for most applications in ([P1]) the characterization $(0.1) - (0.4)$ is sufficient, a weaker version "without continuity constants" is needed for proving that rigidity is well behaved to inductive limits (4.5 in [P1]). For this reason, one introduces in (4.2.2 of [P1]) the notion of ε_0-*rigidity*, which requires that

(0.5). $\exists F_0 \subset N$ finite and $\delta_0 > 0$ such that if ϕ is a completely positive map on N with $\phi(1) \leq 1$, $\tau \circ \phi \leq \tau$ and $\|\phi(x) - x\|_2 \leq \delta_0, \forall x \in F_0$ then $\|\phi(b) - b\|_2 \leq \varepsilon_0, \forall b \in B, \|b\| \leq 1,$

and one proves in (4.3 of [P1]) that if N is a factor and B is regular in N, i.e., $\mathcal{N}_N(B) \overset{\text{def}}{=} \{u \in \mathcal{U}(N) \mid uBu^* = B\}$ generates N, then $B \subset N$ is rigid if and only if it is $1/3$-*rigid*.

But while enough for proving (4.5 in [P1]), ε_0-rigidity is not the exact analogue of the original definition of relative property (T) for groups "without continuity constants", as considered in ([M]).

The purpose of this paper is to provide such an analogue. Thus, we prove a characterization of the rigidity for inclusions of finite von Neumann algebras $B \subset N$ which no longer requires the invariant vector ξ_0 in (0.1) to be close to the almost invariant vector

ξ, but merely to be "almost tracial" from left and right (see conditions (1.2), (1.2$'$) in Theorem 1). This almost traciality condition, which is irrelevant in the group case, is unavoidable in the framework of von Neumann algebras, because of the requirement that the rigidity of an inclusion $B \subset N$ be preserved under reduction by projections in B and $B' \cap N$ (cf. 4.7 in [P1]).

The almost traciality condition is in fact redundant if we assume N is a factor, $B' \cap N \subset B$ and $\mathcal{N}_N(B)' \cap N = \mathbb{C}$ (see Corollary 2). This assumption is the same as the one needed for proving the equivalence between ε_0-rigidity (condition (0.5)) and rigidity (conditions (0.1) – (0.4)) in (4.3 of [P1]). Consequently, we obtain a new proof of that result, which is more conceptual and avoids the use of ultrapower algebras.

Theorem 1. *Let N be a finite von Neumann algebra with countable decomposable center and $B \subset N$ a von Neumann subalgebra. The following conditions are equivalent:*

(1.1). *The inclusion $B \subset N$ is rigid in the sense of Definition* 4.2 *in* [Po1]*, i.e., it satisfies the equivalent conditions* (0.1) – (0.4)*.*

(1.2). *There exists a normal faithful tracial state τ on N with the property: $\exists F_0 \subset N$ finite and $\delta_0 > 0$ such that $\forall \varepsilon > 0$, $\exists \delta(\varepsilon) > 0$ so that if $\mathcal{H}$ is a Hilbert N-bimodule with a vector $\xi \in \mathcal{H}$ satisfying $\|y\xi - \xi y\| \leq \delta_0, \forall y \in F_0$, $\|\langle \cdot \xi, \xi\rangle - \tau\| < \delta, \|\langle \xi \cdot, \xi\rangle - \tau\| < \delta$, then $\exists \xi_0 \in \mathcal{H}$ such that $b\xi_0 = \xi_0 b, \forall b \in B$, $\|\langle \cdot \xi_0, \xi_0\rangle - \tau\| < \varepsilon, \|\langle \xi_0 \cdot, \xi_0\rangle - \tau\| < \varepsilon$.*

(1.2'). *There exists a normal faithful tracial state τ on N with the property: $\forall \varepsilon_0 > 0$, $\exists F_0 = F_0(\varepsilon_0) \subset N$ finite and $\delta_0 = \delta_0(\varepsilon_0) > 0$ such that if $\mathcal{H}$ is a Hilbert N-bimodule with a vector $\xi \in \mathcal{H}$ satisfying $\|y\xi - \xi y\| \leq \delta_0, \forall y \in F_0$, $\|\langle \cdot \xi, \xi\rangle - \tau\| < \delta_0, \|\langle \xi \cdot, \xi\rangle - \tau\| < \delta_0$, then $\exists \xi_0 \in \mathcal{H}$ such that $b\xi_0 = \xi_0 b, \forall b \in B$, $\|\langle \cdot \xi_0, \xi_0\rangle - \tau\| < \varepsilon_0, \|\langle \xi_0 \cdot, \xi_0\rangle - \tau\| < \varepsilon_0$.*

(1.3). *Condition* (1.2) *above is satisfied for any normal faithful tracial state τ on N.*

(1.3'). *Condition* (1.2$'$) *above is satisfied for any normal faithful tracial state τ on N.*

Corollary 2. *Let N be a finite factor and $B \subset N$ a von Neumann subalgebra such that $B' \cap N \subset B$, $\mathcal{N}_N(B)' \cap N = \mathbb{C}$. The following conditions are equivalent:*

(2.1). *The inclusion $B \subset N$ is rigid.*

(2.2). *The inclusion $B \subset N$ is ε_0-rigid for some $0 < \varepsilon_0 < 1$ (i.e., it satisfies condition* (0.5) *for that ε_0).*

(2.3). *$\exists F_0 \subset N$ finite and $\delta_0 > 0$ such that if $\mathcal{H}$ is a Hilbert N-bimodule with a unit vector $\xi \in \mathcal{H}$ satisfying $\|y\xi - \xi y\| \leq \delta_0, \forall y \in F_0$, $\|\langle \cdot \xi, \xi\rangle - \tau\| \leq \delta_0$, $\|\langle \xi, \xi \cdot\rangle - \tau\| \leq \delta_0$, then $\exists \xi_0 \in \mathcal{H}$, $\xi_0 \neq 0$, such that $b\xi_0 = \xi_0 b, \forall b \in B$.*

The proof of Theorem 1 proceeds by contradiction, assuming (1.2$'$) is satisfied while (1.1) is not. By using the point of view of correspondences (i.e., going back and forth from c.p. maps to Hilbet bimodules [C2]) and technical background from ([P1]), this allows us to construct a sequence of completely positive, subunital, subtracial maps ϕ_n

on N that get closer and closer to id_N in point $\|\cdot\|_2$-topology, in a way that makes the infinite product (composition) $\phi = \Pi_n \phi_n$ be well defined, close to id_N on prescribed finite subsets of N and with a "controlled divergence" from id_B, when restricted to B. Moreover, we show that the c.p. maps ϕ_n can be taken so that the operators they induce on $L^2(N, \tau)$ are positive. If $(\mathcal{H}_\phi, \xi_\phi)$ is the pointed Hilbert N-bimodule associated with ϕ as in (1.1 of [P1]), then ξ_ϕ almost commutes with N. Thus, by (1.2′), $\mathcal{H}_\phi$ contains a non-zero B-central vector ξ_0. Approximating ξ_0 by $\Sigma_j x_j \xi_\phi y_j \in \text{sp} N \xi_\phi N$, and using the infinite product form of ϕ, as well as the positivity (as operators) of the ϕ_n's, leads to a contradiction.

We mention that Theorem 1 doesn't directly entail the analogous result for groups in ([Jo]). However, when translating its proof to the case of inclusions of groups, we obtain a more direct and shorter proof of ([Jo]), which we present in the Appendix to this paper. Nevertheless, part of the proof of Theorem 1 was inspired by ([Jo]). Thus, our idea of using infinite products of c.p. maps was triggered by an effort to bypass negative definite functions (for which no satisfactory operator algebra analogue exists) and their infinite sums, used in ([Jo]). For more on infinite products of correspondences, both as c.p. maps and as Hilbert bimodules, see ([Pe]).

Proof of Theorem 1. We clearly have $(1.3') \implies (1.3) \implies (1.2)$ and $(1.3') \implies (1.2') \implies (1.2)$. Also, condition (0.3) implies (1.3′) above, showing that $(1.1) \implies (1.3')$. Thus, in order to finish the proof of the theorem it is sufficient to prove that if (1.2) is satisfied for a certain normal faithful tracial state τ then condition (0.2) is satisfied for that τ.

To this end, we first reduce this implication to the case N is separable. Thus, we begin by proving that (1.2) implies B is separable (in the norm $\| \quad \|_2$ given by τ).

Let $F = \cup_n F_0(1/n)$ and denote by N_0 the von Neumann algebra generated by F, which thus follows separable. Denote $\mathcal{H} = L^2(\langle N, N_0 \rangle, Tr)$ and $\xi = e_{N_0}$. Then $\mathcal{H}$ is a $\langle N, N_0 \rangle$ Hilbert bimodule, so in particular it is a N bimodule. Since $[N_0, \xi] = 0$ and ξ is tracial, by (1.2) it follows that there exists a unit vector $\xi_0 \in \mathcal{H}$ such that $[B, \xi_0] = 0$ and $Tr(\cdot \xi_0 \xi_0^*)$ close to τ_N. It follows that B commutes with $\xi_0 \xi_0^* \in L^1(\langle N, N_0 \rangle, Tr)$, so it also commutes with the spectral projections of $\xi_0 \xi_0^*$. If B is non-separable, then there exists $z_0 \in \mathcal{P}(\mathcal{Z}(B))$ such that Bz is non-separable, $\forall z \in \mathcal{P}(\mathcal{Z}(B))$, $z \leq z_0$. By the condition $Tr(\cdot \xi_0 \xi_0^*)$ close to τ_N, it follows that there exists a spectral projection e of $\xi_0 \xi_0^*$ such that $z_0 e \neq 0$. Thus, $z_0 e \in \langle N, N_0 \rangle$ commutes with B and has finite trace Tr. By (2.2 in [P2]), it follows that for some $p \in \mathcal{P}(Bz_0)$ and $\eta \in \mathcal{H}$ with $p\eta \neq 0$ we have $pBp\eta \subset \overline{\eta N_0}$. But this gives a contradiction, since the closure of ηN_0 in $\mathcal{H}$ is a separable Hilbert space while $pBp\eta$ is not.

Now note that if (1.2) holds true for $(B \subset N, \tau)$ then it holds true for $(B \subset N_1, \tau)$, where N_1 is the (separable) von Neumann algebra generated by B and F. Indeed, this is immediate to see by inducing N_1-bimodules to N-bimodules. Thus, if we assume $(1.2) \implies (0.2)$ for inclusions of separable von Neumann algebras, then it follows that

$B \subset N_1$ satisfies the equivalent conditions (0.1) – (0.4). But then (4.6 in [P1]) shows that $B \subset N$ satisfies these conditions as well.

Thus, from now on we may assume N is separable. We need the following equivalent characterization of (0.2).

Lemma 3. *Condition* (0.2) *for* $(B \subset N, \tau)$ *holds true if and only if it holds true for completely positive maps* ϕ *with the property* $T_\phi \geq 0$.

Proof. We first prove that if $(B \subset N, \tau)$ satifies:

(3.1). $\forall \varepsilon > 0$, $\exists F' = F'(\varepsilon) \subset \mathcal{U}(N)$ finite and $\delta' = \delta'(\varepsilon) > 0$ such that if $\phi : N \to N$ is a normal, completely positive map with $\phi^* = \phi$, $\tau \circ \phi \leq \tau, \phi(1) \leq 1$ and $\|\phi(x) - x\|_2 \leq \delta', \forall x \in F'$, then $\|\phi(b) - b\|_2 \leq \varepsilon, \forall b \in B$, $\|b\| \leq 1$.

then $(B \subset N, \tau), F''(\varepsilon) = F'(\varepsilon^2/2)$, $\delta''(\varepsilon) = \delta'(\varepsilon^2/2)^2/2$ satisfy the condition:

(3.2). If $\phi : N \to N$ is a normal, completely positive map with $\tau \circ \phi \leq \tau, \phi(1) \leq 1$ and $\|\phi(x) - x\|_2 \leq \delta'', \forall x \in F''$, then $\|\phi(b) - b\|_2 \leq \varepsilon, \forall b \in B$, $\|b\| \leq 1$.

Let ϕ be as in the hypothesis of (3.2). Then $\phi' = (\phi + \phi^*)/2$ satisfies $T^*_{\phi'} = T_{\phi'}$ while we still have $\phi'(1) \leq 1$, $\tau \circ \phi' \leq \tau$. By (1.1.5.3° in [P1]), it follows that if $x \in F''$ then

$$\|\phi'(x) - x\|_2 \leq (\|\phi(x) - x\|_2 + \|\phi^*(x) - x\|_2)/2$$

$$\leq (\|\phi(x) - x\|_2 + (2\|\phi(x) - x\|_2)^{1/2})/2 \leq (\delta'' + (2\delta'')^{1/2})/2 \leq \delta'(\varepsilon^2/2)$$

By (3.1), the above inequality implies that $\|\phi'(b) - b\|_2 \leq \varepsilon^2/2$, $\forall b \in B, \|b\| \leq 1$. But then we also have for $b \in B, \|b\| \leq 1$ the estimates:

$$\|\phi(b) - b\|_2^2 \leq 2\tau(bb^*) - 2\mathrm{Re}\tau(\phi(b)b^*)$$

$$= 2\tau(bb^*) - 2\mathrm{Re}\tau(\phi'(b)b^*) \leq 2\|b - \phi'(b)\|_2 \leq \varepsilon^2,$$

showing that $\|\phi(b) - b\|_2 \leq \varepsilon$, $\forall b \in B, \|b\| \leq 1$.

Thus, in order to prove Lemma 3 it is now sufficient to show that if

(3.0). $\forall \varepsilon > 0$, $\exists F_0 = F_0(\varepsilon) \subset \mathcal{U}(N)$ finite and $\delta_0 = \delta_0(\varepsilon) > 0$ such that if $\phi : N \to N$ is a normal, completely positive map with $T_\phi \geq 0$, $\tau \circ \phi \leq \tau, \phi(1) \leq 1$ and $\|\phi(x) - x\|_2 \leq \delta_0, \forall x \in F_0$, then $\|\phi(b) - b\|_2 \leq \varepsilon, \forall b \in B$, $\|b\| \leq 1$.

then $(B \subset N, \tau), F_1(\varepsilon) = F_0(\varepsilon/e^2)$, $\delta_1(\varepsilon) = \delta_0(\varepsilon/e^2)/4$ satisfy the condition:

(3.1′). If $\phi : N \to N$ is a normal, completely positive map with $\tau \circ \phi \leq \tau, \phi(1) \leq 1$, $\phi^* = \phi$ and $\|\phi(x) - x\|_2 \leq \delta_1, \forall x \in F_1$, then $\|\phi(b) - b\|_2 \leq \varepsilon, \forall b \in B$, $\|b\| \leq 1$.

To show this, let $\phi : N \to N$ be a c.p. map as in (3.1′). Define ϕ'' on N by $\phi'' = exp(\phi - id_N) = e^{-1}exp(\phi) = e^{-1}\Sigma_{n=0}^{\infty}\phi^n/n!$, where ϕ^n denotes the n time

composition $\phi \circ ... \circ \phi$. By the definition, it follows that ϕ'' is completely positive. Also, since $T_\phi = T_\phi{}^*$ and $T_{\phi''} = e^{-1}exp(T_\phi)$, it follows that $T_{\phi''}$ is a positive operator on the Hilbert space $L^2(N, \tau)$. Moreover, since $\tau \circ \phi \leq \tau$, we have $\tau \circ \phi^n \leq \tau, \forall n$, and thus

$$\tau \circ \phi'' = e^{-1}\Sigma_n \tau \circ \phi^n/n! \leq \tau \circ \phi \leq \tau.$$

Similarly, $\phi(1) \leq 1$ implies

$$\phi''(1) = e^{-1}\Sigma_n \phi^n(1)/n! \leq \phi(1) \leq 1.$$

By taking into account that

$$\|\phi''(x) - x\|_2 = \|\Sigma_{n=0}^{\infty}(\phi - id_N)^n(x)/n! - x\|_2 = \|\Sigma_{n=1}^{\infty}(\phi - id_N)^n(x)/n!\|_2$$

$$\leq (\Sigma_{n=1}^{\infty} 2^{n-1}/n!)\|\phi(x) - x\|_2 = ((e^2 - 1)/2)\|\phi(x) - x\|_2 < 4\|\phi(x) - x\|_2,$$

it follows that if ϕ satisfies the hypothesis of $(3.1')$ then ϕ'' satisfies the conditions in (3.0) for ε of the form $e^{-2}\varepsilon$. Thus, by (3.0) we have $\|\phi''(b) - b\|_2 \leq e^{-2}\varepsilon$, $\forall b \in B, \|b\| \leq 1$. To obtain from this that $\|\phi(b) - b\|_2$ is uniformly small for b in the unit ball of B, denote $f(t) = 1 - e^{-t}, 0 \leq t \leq 2$. Since $f(0) = 0, f'(t) = e^{-t} \geq e^{-2}$, it follows that $f(t) \geq te^{-2}, \forall 0 \leq t \leq 2$. Hence, if we let $S = 1 - T_\phi \in \mathcal{B}(L^2(N, \tau))$ then $0 \leq S \leq 2$, which by functional calculus gives $f(S) \geq e^{-2}S$. We thus get the estimates:

$$e^{-2}\varepsilon \geq \|\phi''(b) - b\|_2 = \|f(S)(\hat{b})\|_2$$

$$\geq e^{-2}\|S(\hat{b})\|_2 = e^{-2}\|b - \phi(b)\|_2$$

for all b in B with $\|b\| \leq 1$.

$\square$

Proof of (1.2) $\implies$ (0.2). We proceed by contradiction, assuming that (1.2) holds true while (0.2) doesn't. Note first that "non-(0.2)" and the separability of N implies:

$\overline{(0.2)}$. There exist $c_0 > 0$, c.p. maps $\{\phi_n\}_n$ on N and unitary elements $\{b_n\}_n \subset \mathcal{U}(B)$, such that $T_{\phi_n} \geq 0$, $\tau \circ \phi_n \leq \tau$, $\phi_n(1) \leq 1$, $\|\phi_n(x) - x\|_2 \to 0$, $\forall x \in N$, and $\|\phi_n(b_n) - b_n\|_2 \geq c_0, \forall n$.

Note also that the inequality $\|\phi_n(b_n) - b_n\|_2 \geq c_0$ implies

$$c_0^2 \leq \|\phi_n(b_n) - b_n\|_2^2 \leq 2 - 2\tau(\phi_n(b_n)b_n^*)$$

which in turn gives

$$\|\phi_n(b_n)\|_2^2 = \tau(\phi_n^2(b_n)b_n^*) \leq \tau(\phi_n(b_n)b_n^*) \leq 1 - c_0^2/2.$$

Thus, if we let $c_1 = (1 - c_0^2/2)^{1/2} < 1$, then $\overline{(0.2)}$ implies:

$\overline{(3.2)}$. There exist $c_1 < 1$, completely positive maps $\{\phi_n\}_n$ on N and $\{b_n\}_n \in \mathcal{U}(B)$, such that T_{ϕ_n} are positive operators on $L^2(N, \tau)$, $\tau \circ \phi_n \leq \tau$, $\phi_n(1) \leq 1$, $\|\phi_n(x) - x\|_2 \to 0$, $\forall x \in N$, and $\|\phi_n(b_n)\|_2 \leq c_1, \forall n$.

From this point on, we'll need the following notation: If $Y = \{(y_j, z_j)\}_j \subset N \times N$ is a finite set and $\xi \in \mathcal{H}$ for some N bimodule $\mathcal{H}$ then we denote $\mathcal{T}_Y(\xi) = \Sigma_j y_j \xi z_j$. Also, if $\phi : N \to N$ is a linear map then $\mathcal{T}_Y \phi(x) = \Sigma_j z_j^* \phi(y_j^* x y_i) z_i$. Note that if ϕ is the normal c.p. map on N given by a vector "bounded from the right" ξ in a Hilbert N-bimodule $\mathcal{H}$, as in (1.1.3 of [P1]), then $\mathcal{T}_Y \phi$ is the normal c.p. map given by the vector $\mathcal{T}_Y(\xi)$.

Let $\varepsilon > 0$ be so that $\varepsilon < (1 - c_1)/(1 + c_1)$, where c_1 is as given by $\overline{(3.2)}$. Let F_0, $\delta_0, \delta = \delta(\varepsilon^2/16)$ be as given by (1.2), with $1 \in F_0$. Choose a countable $\mathbb{Q} + i\mathbb{Q}$ $\| \quad \|_2$-dense subalgebra $N_0 = \{x_n\}_n \subset N$. Denote by $\{Y_n\}_n$ the collection of all finite sets of pairs of elements in N_0 (i.e., $\{Y_n\}_n$ is the set of finite subsets of $N_0 \times N_0$). Let $\{\phi_n\}_n$ be the c.p. maps given by $\overline{(3.2)}$. We choose recursively an increasing sequence of finite sets $S_k \subset N$ and an increasing sequence of integers n_k, $k \geq 0$, such that if we set $S_0 = F_0$, $n_0 = 0, \phi_0 = id_N$, $Y_0 = \{(1, 1)\}$ then for all $k \geq 1$:

(a) $$\{b_{n_i} \mid 1 \leq i \leq k - 1\}, \{x_j \mid 1 \leq j \leq k\} \subset S_k$$

(b) $$\mathcal{T}_{Y_j}(S_{k-1}) \subset S_k, \forall j \leq k$$

(c) $$(\phi_{n_{k-1}} \circ ... \circ \phi_{n_j} \circ \mathcal{T}_{Y_i}(\phi_{n_{j-1}} \circ ... \circ \phi_{n_1}))(S_{k-1}) \subset S_k, \forall j \leq k - 1, i \leq k$$

(d) $$\|\phi_{n_k}(x) - x\|_2 \leq \delta_0^2/2^{k+4}, \forall x \in S_k, \|\phi_{n_k}(1) - 1\|_2 \leq \delta^2/2^{k+4}$$

(e) $$\|\phi_{n_k}(b_{n_k})\|_2 \leq c_1, \forall k \geq 1$$

By $(a) - (d)$, it follows that $\|\phi_{n_k} \circ ... \circ \phi_{n_1}(x_j) - \phi_{n_{k-1}} \circ ... \circ \phi_{n_1}(x_j)\|_2 \leq \delta_0^2 2^{-k-4}$, $\forall j \leq k$. Thus, $\{\phi_{n_k} \circ ... \circ \phi_{n_1}(x_j)\}_k$ is Cauchy for each j. Since $\tau \circ \phi_{n_k} \circ ... \circ \phi_{n_0} \leq \tau$ and $\phi_{n_k} \circ ... \circ \phi_{n_0}(1) \leq 1$, by the density of $\{x_j\}_j$ in N it follows that in fact $\{\phi_{n_k} \circ ... \circ \phi_{n_1}(x)\}_k$ is Cauchy $\forall x \in N$. Thus, we can define $\phi(x) = \lim_{k \to \infty} \phi_{n_k} \circ ... \circ \phi_{n_0}(x)$, $\forall x \in N$, which follows c.p. on N with $\tau \circ \phi \leq \tau, \phi(1) \leq 1$. Moreover, we have

$$\|\phi(x) - x\|_2 \leq \Sigma_{k=0}^{\infty} \|\phi_{n_k} \circ ... \circ \phi_{n_0}(x) - \phi_{n_{k-1}} \circ ... \circ \phi_{n_0}(x)\|_2$$

$$\leq \delta_0^2 \Sigma_{k=1}^{\infty} 2^{-k-4} = \delta_0^2/16,$$

for all $x \in F_0$. Similarly, $\|\phi(1) - 1\|_2 \leq \delta^2/16$. Thus, if we denote $(\mathcal{H}_\phi, \xi_\phi)$ the pointed Hilbert bimodule associated with $\tau(\phi(1))^{-1}\phi$ then by (1.1.2.4° in [P1]) we have

$$\|x\xi_\phi - \xi_\phi x\|^2 \leq 2\|\tau(\phi(1))^{-1}\phi(x) - x\|_2^2 + 2\|\tau(\phi(1))^{-1}\phi(1)\|_2|\tau(\phi(1))^{-1}\phi(x) - x\|_2$$

$$\leq 2(1 - \delta^2/16)^{-2}((\|\phi(x) - x\|_2 + \|\phi(1) - 1\|_2)^2 + (\|\phi(x) - x\|_2 + \|\phi(1) - 1\|_2))$$
$$\leq 2(1 - \delta^2/16)^{-2} 2(\delta_0^2/16 + \delta^2/16) \leq \delta_0^2,$$

for all $x \in F_0$, whenever $\delta \leq \delta_0 < 1$. Also, since $\tau \circ \phi \leq \tau$, by (1.1.5.3° in [P1]) we get

$$\|\langle \cdot \xi_\phi, \xi_\phi\rangle - \tau\| = \|\phi^*(1) - \tau(\phi^*(1))\|_1 \tau(\phi(1))^{-1}$$

$$2(1 - \delta^2/16)^{-1}\|\phi^*(1) - 1\|_2 \leq 2(1 - \delta^2/16)^{-1}(2\|\phi(1) - 1\|_2)^{1/2}$$
$$< 3(1 - \delta^2/16)^{-1}\delta/4 < \delta.$$

Similarly, we have

$$\|\langle \xi_\phi \cdot, \xi_\phi\rangle - \tau\| = \tau(\phi(1))^{-1}\|\phi(1) - \tau(\phi(1))\|_1$$

$$\leq 2(1 - \delta^2/16)^{-1}\|\phi(1) - 1\|_2 < \delta.$$

Thus, (1.2) applies to get a unit vector $\xi_0 \in \mathcal{H}_\phi$ such that $b\xi_0 = \xi_0 b, \forall b \in B$ and $\|\langle \cdot \xi_0, \xi_0\rangle - \tau\| < \varepsilon^2/16$, $\|\langle \xi_0 \cdot, \xi_0\rangle - \tau\| < \varepsilon^2/16$.

We'll now use the density of the set $\{x_n\}_n$ in N and Kaplansky's theorem to show that there exists a vector of the form $\mathcal{T}_{Y_n}\xi_\phi$ which satisfies the same properties as ξ_0:

Lemma 4. *There exists $Y \subset N_0 \times N_0$ such that $\mathcal{T}_Y\phi$ and $\xi = \mathcal{T}_Y(\xi_\phi)$ satisfy $\|b\xi - \xi b\| \leq \varepsilon$, $\|\mathcal{T}_Y\phi(b) - b\|_2 < \varepsilon$, $\forall b \in \mathcal{U}(B)$, $\|\langle \cdot \xi, \xi\rangle - \tau\| < \varepsilon$, $\|\langle \xi \cdot, \xi\rangle - \tau\| < \varepsilon$.*

Proof. Since $\{\mathcal{T}_{Y_n}(\xi_\phi)\}_n = \mathrm{sp} N_0 \xi_\phi N_0$ is dense in $\mathrm{sp} N \xi_\phi N$ which in turn is dense in $\mathcal{H}_\phi$, it follows that there exists n_0 such that if we denote $Y' = Y_{n_0}$, then $\xi = \mathcal{T}_{Y'}(\xi_\phi)$ is close enough to ξ_0 to ensure that $\|b\xi - \xi b\| \leq \varepsilon^2/16, \forall b \in \mathcal{U}(B)$, while we still have $\|\langle \cdot \xi, \xi\rangle - \tau\| < \varepsilon^2/16$, $\|\langle \xi \cdot, \xi\rangle - \tau\| < \varepsilon^2/16$.

Let $a_0 = \mathcal{T}_Y\phi(1)$, $d_0 = (\mathcal{T}_Y\phi)^*(1)$. Then the last two inequalities become $\|a_0 - 1\|_1 < \varepsilon^2/16$, $\|d_0 - 1\|_1 < \varepsilon^2/16$. Since ξ implements the c.p. map $\mathcal{T}_{Y'}(\phi)$, by (Lemma 1.1.3 in [P1]) it follows that if we let $a = a_0 \vee 1$, $d = d_0 \vee 1$ and $\xi' = a^{-1/2}\xi d^{-1/2}$ then $\|\xi - \xi'\|^2 \leq 8\varepsilon^2/16 = \varepsilon^2/2$.

By Kaplansky's density theorem there exist a'_n, d'_n in the unit ball of N_0 such that $\lim_{n\to\infty} \|a'_n - a^{-1/2}\|_2 = 0$, $\lim_{n\to\infty} \|d'_n - d^{-1/2}\|_2 = 0$.

Denote $Y'_n = \{(a'_n x, y d'_n) \mid (x,y) \in Y'\}$, $\xi'_n = \mathcal{T}_{Y'_n}(\xi_\phi) = a'_n \xi d'_n$, $\phi' = \phi_{\xi'}$, $\phi'_n = \mathcal{T}_{Y'_n}\phi$ and note that $Y'_n \in N_0 \times N_0$, $\forall n$. It follows that $\lim_{n\to\infty} \|\xi' - \xi'_n\| = 0$. Also, it is easy to see that $\|\phi'_n(1)\| \le \|a_0\|$. This implies $\|\phi'_n(1) - \phi'(1)\|_2 \to 0$, so in particular $\|\phi'_n(1)\|_2 \to \|\phi'(1)\|_2$. But by (1.1.3 in [P1]) we have

$$\|\phi'_n(b) - b\|_2^2 \le \|[b, \xi'_n]\|^2 + (\|\phi'_n(1)\|_2^2 - 1)$$

and since

$$\begin{aligned}\|[b,\xi'_n]\| &\le \|[b,\xi]\| + \|\xi - \xi'\| + \|\xi' - \xi'_n\| \\ &\le \varepsilon^2/16 + 2^{-1/2}\varepsilon + \|\xi' - \xi'_n\|,\end{aligned}$$

for large enough n and $\varepsilon < 1$ (to insure that $\varepsilon^2/16 + 2^{-1/2}\varepsilon < \varepsilon$) we obtain the estimate $\|\phi'_n(b) - b\|_2 < \varepsilon, \forall b \in \mathcal{U}(B)$.

Thus, if we put $Y = Y'_n$ then all the requirements are satisfied.

□

For each $1 \le j \le k$, denote $\phi_j^k = \phi_{n_k} \circ ... \circ \phi_{n_j}$, $\phi_j^\infty = \lim_{k\to\infty} \phi_j^k$. Since $\phi_{n_k} \to id_N$, by (Corollary 1.1.2 in [P1]) it follows that $\|\phi_{n_k}(y^* \cdot z) - y^*\phi_{n_k}(\cdot)z\| \to 0$, $\forall y, z \in N$, and thus

(f) $$\lim_{k\to\infty} \|\mathcal{T}_Y\phi - \phi_{k+1}^\infty(\mathcal{T}_Y\phi_1^k)\| = 0$$

Since by $(a),(b),(c)$ we have $\phi_{m+1}^j(\phi_{k+1}^m(\mathcal{T}_Y\phi_1^k(b_{n_m}))) \in S_j$, $\forall j > m$, we also get

(g) $$\lim_{m\to\infty} \|\phi_{k+1}^\infty(\mathcal{T}_Y\phi_1^k)(b_{n_m}) - \phi_{k+1}^m(\mathcal{T}_Y\phi_1^k)(b_{n_m})\|_2 = 0$$

Altogether, from (f) and (g) we obtain:

$$\begin{aligned}&\|b_{n_m} - \phi_{k+1}^m(\mathcal{T}_Y\phi_1^k)(b_{n_m})\|_2 \\ &\le \|b_{n_m} - \phi_{k+1}^\infty(\mathcal{T}_Y\phi_1^k)(b_{n_m})\|_2 + \|\phi_{k+1}^\infty(\mathcal{T}_Y\phi_1^k)(b_{n_m}) - \phi_{k+1}^m(\mathcal{T}_Y\phi_1^k)(b_{n_m})\|_2 \\ &\le \|b_{n_m} - \mathcal{T}_Y\phi(b_{n_m})\|_2 + \|\mathcal{T}_Y\phi - \phi_{k+1}^\infty(\mathcal{T}_Y\phi_1^k)\| + \varepsilon(m) \\ &\le \|b_{n_m} - \mathcal{T}_Y\phi(b_{n_m})\|_2 + \varepsilon'(k) + \varepsilon(m) \\ &< \varepsilon + \varepsilon'(k) + \varepsilon(m)\end{aligned}$$

for all $m > k$, with $\varepsilon(m), \varepsilon'(k)$ satisfying $\lim_{m\to\infty} \varepsilon(m) = 0$, $\lim_{k\to\infty} \varepsilon'(k) = 0$. Thus, there exists k_0 such that for all $m > k \ge k_0$ we have

(h) $$\|b_{n_m} - \phi_{k+1}^m(\mathcal{T}_Y\phi_1^k)(b_{n_m})\|_2 \le \varepsilon$$

Moreover, since by Lemma 4 we have $\|\mathcal{T}_Y\phi(1)\|_2 < 1+\varepsilon$, by (f) it follows that we can choose k_0 so that for all $k \geq k_0$ we also have

$$\text{(i)} \qquad \|\phi_{k+1}^{\infty}(\mathcal{T}_Y\phi_1^k)(1)\|_2 < 1+\varepsilon$$

Fix some $k \geq k_0$. By (g) and (i) it follows that there exists $m > k$ such that

$$\text{(j)} \qquad \|\phi_{k+1}^{m-1}(\mathcal{T}_Y\phi_1^k)(1)\|_2 < 1+\varepsilon$$

On the other hand, since b_{n_m} are unitary elements, by (part 1° in Lemma 1.1.2 of [P1]) we have

$$\text{(k)} \qquad \|\phi_{k+1}^{m-1}(\mathcal{T}_Y\phi_1^k)(b_{n_m})\|_2 \leq \|\phi_{k+1}^{m-1}(\mathcal{T}_Y\phi_1^k)(1)\|_2$$

Combining (j) and (k) we get

$$\text{(l)} \qquad \|\phi_{k+1}^{m-1}(\mathcal{T}_Y\phi_1^k)(b_{n_m})\|_2 \leq 1+\varepsilon$$

For simplicity, denote by T the operator implemented by $\phi_{k+1}^{m-1}(\mathcal{T}_Y\phi_1^k)$ on $L^2(N,\tau)$, by S the operator implemented by $\phi_{n_m} = \phi_m^m$ on $L^2(N,\tau)$ and by η the vector $\hat{b}_{n_m}$. Recall that $S \geq 0$ (so in particular $S = S^*$). By (e) we have $\|S(\eta)\|_2 \leq c_1$ and by (l) we have $\|T(\eta)\|_2 \leq 1+\varepsilon$. Also, by (h) we have $\|\eta - ST(\eta)\|_2 \leq \varepsilon$. By applying twice the Cauchy-Schwartz inequality, it follows that

$$\varepsilon \geq |\langle \eta - ST(\eta), \eta\rangle|$$

$$= |1 - \langle T(\eta), S(\eta)\rangle| \geq 1 - \|T(\eta)\|_2\|S(\eta)\|_2 \geq 1 - c_1(1+\varepsilon)$$

But this implies $\varepsilon \geq (1-c_1)/(1+c_1)$, in contradiction with our initial choice of ε.

□

Proof of Corollary 2. We clearly have (2.1) $\implies$ (2.2). To prove (2.2) $\implies$ (2.3), let $(\mathcal{H}, \xi)$ be a pointed Hilbert N-bimodule satisfying (2.3) for some $F_0 \subset N$ finite $\delta_0 > 0$. Denote $a_0, b_0 \in L^1(N,\tau)$ the Radon-Nykodim derivatives of $\langle \cdot\xi, \xi\rangle$ and $\langle \xi\cdot, \xi\rangle$ with respect to τ. Let $\xi' = (a_0 \vee 1) - 1/2\xi(b_0 \vee 1)^{-1/2}$ and note that ξ' implements subtracial functionals on N, both left and right. By (1.1.3.1° in [P1]) we have $\|\xi - \xi'\|^2 \leq 8\delta_0$. Also, since $\|(a_0 \vee 1)^{-1}\xi\| \leq 1$, $\xi(b_0 \vee 1)^{-1}\| \leq 1$, by applying the Cauchy-Schwartz inequality we get

$$\|\xi'\|^2 = \langle (a_0 \vee 1)^{-1}\xi, \xi(b_0 \vee 1)^{-1}\rangle$$

$$\geq \langle \xi, \xi\rangle - \|(a_0 \vee 1)^{-1}\xi - \xi\| - \|\xi - \xi(b_0 \vee 1)^{-1}\|$$

$$\geq 1 - (\tau((1-(a_0 \vee 1)^{-1})^2) + \|a_0 - 1\|_1)^{1/2} - (\tau((1-(b_0 \vee 1)^{-1})^2) + \|b_0 - 1\|_1)^{1/2}$$

$$\geq 1-(2\|a_0-1\|_1)^{1/2}+(2\|b_0-1\|_1)^{1/2}\geq 1-3\delta_0^{1/2}.$$

Denote $\phi'=\phi_{(\mathcal{H},\xi'')}$ the c.p. map associated with $\xi''=\|\xi'\|^{-1}\xi'$ as in (1.1.3 in [P1]). By (1.1.3.1° in [P1]) we have $\|\xi-\xi''\|^2\leq 8(1-3\delta_0^{1/2})^{-1}\delta_0$. Also, by (1.1.3.2° in [P1]), if we take $\delta_0\leq 1/16$ then we have:

$$\|\phi'(u)-u\|_2\leq \|[u,\xi'']\|^2+(\|\phi'(1)\|_2^2-1)$$

$$\leq \|[u,\xi']\|^2+(1-3\delta_0^{1/2})^{-2}-1$$

$$\leq (\|[u,\xi]\|+\|\xi-\xi'\|)^2+100\delta_0^{1/2},$$

for all unitary elements in N, and thus for all x in the unit ball of N. Thus, if we take δ_0 sufficiently small and apply the inequality to $x\in F_0$, then ϕ' checks condition (2.2). Thus, $\|\phi'(b)-b\|_2\leq\varepsilon_0,\forall b\in\mathcal{U}(B)$. But then we have the estimates

$$\varepsilon_0\geq|\langle b-\phi'(b),b\rangle|=|1-\tau(\phi'(b)b^*)|$$

$$=|\langle\xi'',(\xi''-b\xi''b^*)\rangle|.$$

Thus, if we let ξ_0 be element of minimal norm in $\overline{\text{co}}\{b\xi''b^*\mid b\in\mathcal{U}(B)\}\subset\mathcal{H}$ then $\xi_0\neq 0$ and $[\xi_0,B]=0$. This ends the proof of (2.2) $\implies$ (2.3).

By Theorem 1, in order to prove (2.3) $\implies$ (2.1) it is sufficient to show that if $\mathcal{H}$ is a Hilbert N-bimodule with a non-zero B-central vector $\xi_0\in\mathcal{H}$, then $\mathcal{H}$ contains a non-zero B-central vector $\xi_1\in\mathcal{H}$ which is left and right almost tracial on N. To see this, let $a_0,b_0\in L^1(N,\tau)_+$ be so that $\tau(xa_0)=\langle x\xi_0,\xi_0\rangle,\forall x\in N$ and $\tau(b_0x)=\langle\xi_0x,\xi_0\rangle,\forall x\in N$. Since $[B,\xi_0]=0$ and $B'\cap N=\mathcal{Z}(B)$, both a_0,b_0 belong to $L^1(\mathcal{Z}(B),\tau)_+$ and have the same support projection.

Thus, given any $\varepsilon>0$ there exists a non-zero projection $q\in\mathcal{Z}(B)$ such that $\|a_0q-cq\|<\varepsilon$, $\|b_0q-cq\|<\varepsilon$, where $c=\langle q\xi_0,\xi_0\rangle/\tau(q)=\tau(a_0q)/tau(q)=\tau(b_0q)/\tau(q)$. Moreover, since $\mathcal{N}(B)$ acts ergodically on $\mathcal{Z}(B)$ (because $\mathcal{N}(B)'\cap'\cap N=\mathbb{C}$), it follows that we can take q to satisfy $\tau(q)=1/n$, for some integer $n\geq 1$. Furthermore, there exist partial isometries $v_j\in\mathcal{GN}(B)$, $1\leq j\leq n$, such that $v_j^*v_j=q,\forall j$, $\Sigma_j v_jv_j^*=1$. But then $\xi_1=c^{-1/2}\Sigma_j v_j\xi_0v_j^*$ is easily seen to satisfy $[\xi_1,B]=0$, $\|\langle\cdot\xi_1,\xi_1\rangle-\tau\|\leq\varepsilon$, $\|\langle\xi_1\cdot,\xi_1\rangle-\tau\|\leq\varepsilon$.

$\square$

Remarks 5. 1°. The equivalence in Corollary 2 can actually be proved without assuming $B'\cap N\subset B$, but the argument becomes considerably longer.

2°. Another proof of Lemma 3 can be obtained by following the argument on page (40 of [P1]), which shows that if $\{\phi_n\}_n$ are the c.p. maps given by $\overline{(0.2)}$ then for "large" k and "very large" n the average map $\psi=k^{-1}\Sigma_{j=1}^k\phi_n^j$ shrinks the norm $\|\cdot\|_2$ of some elements b in the unit ball of B by a uniform constant $c_1<1$. But then $\phi=\psi^*\circ\psi$

still shrinks b by c_1 (thus also some unitaries in B, by the Russo-Dye Theorem) and is positive as an operator on $L^2(N,\tau)$.

3°. By (4.7.2° in [P1]), if an inclusion of finite von Neumann algebras with countable decomposable center $B \subset N$ satisfies any of the equivalent conditions in Theorem 1 then it satisfies the following *uniform local weak rigidity* condition (u.l.w.r.): "For all projections $p \in \mathcal{P}(B) \cup \mathcal{P}(B' \cap N)$ the inclusion $pBp \subset pNp$ satifies condition (2.3)." It would be interesting to prove that, conversely, if $B \subset N$ satisfies the u.l.w.r. condition then it is rigid.

Appendix

We present here a short proof of the result in ([Jo]) showing that the original definition of the relative property (T) for inclusions of groups $H \subset G$ in ([M]) is equivalent to a condition "with continuity constants", which can be formulated both in terms of representations and positive definite functions. Our proof uses infinite products of positive definite functions, and is obtained by translating the proof of Theorem 1 to the case of groups, where many simplifications occur. For simplicity, we present the proof for discrete groups only, noting that the same argument works for locally compact groups.

Theorem [Jo]. *Let G be a countable discrete group and $H \subset G$ a subgroup. The following conditions are equivalent:*

$(A.1)$. $\forall \varepsilon > 0$, $\exists F = F(\varepsilon) \subset G$ finite and $\delta = \delta(\varepsilon) > 0$ so that if $\varphi : G \to \mathbb{C}$ is a positive definite function and $|\varphi(g) - 1| \leq \delta, \forall g \in F$ then $|\varphi(h) - 1| \leq \varepsilon, \forall h \in H$.

$(A.2)$. $\forall \varepsilon > 0$, $\exists F' = F'(\varepsilon) \subset G$ finite and $\delta' = \delta'(\varepsilon) > 0$ so that if $\pi : G \to \mathcal{U}(\mathcal{H})$ is a unitary representation of G with a unit vector $\xi \in \mathcal{H}$ satisfying $\|\pi(g)\xi - \xi\| \leq \delta', \forall g \in F'$ then $\exists \xi_0 \in \mathcal{H}$ such that $\pi(h)\xi_0 = \xi_0, \forall h \in H$ and $\|\xi_0 - \xi\| \leq \varepsilon$.

$(A.3)$. $\exists F_0 \subset G$ finite and $\delta_0 > 0$ so that if $\pi : G \to \mathcal{U}(\mathcal{H})$ is a unitary representation of G with a unit vector $\xi \in \mathcal{H}$ satisfying $\|\pi(g)\xi - \xi\| \leq \delta_0, \forall g \in F_0$ then $\exists \xi_0 \in \mathcal{H}$, $\|\xi_0\| = 1$ such that $\pi(h)\xi_0 = \xi_0, \forall h \in H$.

Proof.$(A.1) \implies (A.2)$ is easily seen using the GNS construction and $(A.2) \implies (A.3)$ is trivial hence we will proceed with $(A.3) \implies (A.1)$.

Assume that $(A.3)$ holds but $(A.1)$ does not. Since $(A.1)$ does not hold we have

$\overline{(A.1)}$. $\exists c_0 > 0$, $\{b_n\}_n$ a sequence in G, and $\varphi_n : G \to \mathbb{C}$ a sequence of positive definite functions such that $|\varphi_n(g) - 1| \to 0, \forall g \in G$, and $|\varphi(b_n) - 1| \geq c_0, \forall n$.

Moreover, as $\varphi_n(1) \to 1$, by substituting $\varphi_n(1)^{-1}\varphi_n(g)$ for φ_n we may assume $\varphi_n(1) = 1, \forall n$. Also, since $\varphi'_n(g) = exp(Re(\varphi_n(g)) - 1)$ satisfies $|\varphi'_n(g) - 1| \to 0, \forall g \in G$ and $1 - \varphi'_n(b_n) \geq e^{-2}(1 - Re(\varphi_n(b_n))) \geq e^{-2}/2|\varphi_n(b_n) - 1|^2 \geq e^{-2}/2c_0^2$ (the latter due

to the inequality $1-e^{-t} \geq e^{-2}t, \forall 0 \leq t \leq 2$), by setting $c_1 = e^{-2}/2c_0^2$ and substituting φ_n' for φ_n it follows that $\overline{(A.1)}$ implies:

$\overline{(A.1)'}$. $\exists c_1 > 0$, $\{b_n\}_n$ a sequence in G, and $\varphi_n : G \to \mathbb{C}$ a sequence of positive definite functions such that $|\varphi_n(g)-1| \to 0, \forall g \in G$, $\varphi_n(1) = 1, \varphi_n \geq 0$ and $|\varphi(b_n)-1| \geq c_1, \forall n$.

Note that since $\varphi_n(g) \leq \varphi_n(1) = 1$ we have $\varphi_n(b_n) \leq 1 - c_1, \forall n$. Also if $\{K_n\}_n$ is an increasing sequence of finite sets in G such that $\cup K_n = G$ then since $(\varphi_n(b_n))^k \leq (1-c_1)^k, \forall n$ we may construct a new sequence of positive definite functions by letting $\varphi_k' = (\varphi_{n_k})^k, \forall k$, where $n_k >> n_{k-1}$ are chosen increasing rapidly enough to ensure that $|(\varphi_{n_k}(g))^k - 1| \leq 1/k, \forall g \in K_k$. We then have that φ_k' still satisfy $\overline{(A.1')}$ but also if $b_k' = b_{n_k}$ then $\varphi_k'(b_k') \to 0$.

If $Y = \{(\alpha_j, g_j)\}_j \subset \mathbb{C} \times G$ is a finite set and $\pi : G \to \mathcal{U}(\mathcal{H})$ is a unitary representation with $\xi \in \mathcal{H}$ then we denote $T_Y(\xi) \overset{\text{def}}{=} \Sigma_j \alpha_j \pi(g_j)\xi$. Also, if $\psi : G \to \mathbb{C}$ is a function then $T_Y\varphi(g) \overset{\text{def}}{=} \Sigma_{i,j}\overline{\alpha_j}\alpha_i\varphi(g_j^{-1}gg_i)$. Note that if ψ is the positive definite function on G given by a vector ξ in $\mathcal{H}$, then $T_Y\psi$ is the positive definite function given by the vector $T_Y(\xi)$. Also note that $T_Y(\psi_1\psi_2)(g) = \Sigma_{i,j}\overline{\alpha_j}\alpha_i\psi_1(g_j^{-1}gg_i)\psi_2(g_j^{-1}gg_i)$, which is different from $\psi_1 T_Y(\psi_2)$ and $T_Y(\psi_1)T_Y(\psi_2)$.

Let $\varepsilon > 0$. Let F_0, δ_0 be as given by $(A.3)$. Let $\{K_k\}_k$ be an increasing sequence of finite sets in G such that $F_0 \subset K_1$, and $\cup_k K_k = G$. Let $\{\varphi_k'\}_k$ be the c.p. maps given above. We choose a subsequence $\{\varphi_{k_j}'\}_j$ of $\{\varphi_k'\}_k$ such that:

(a) $$|\varphi_{k_j}'(g) - 1| \leq \delta_0^2/2^{j+1}, \forall g \in K_j$$

By (a), it follows that $|\Pi_{i=1}^j \varphi_{k_i}'(g) - \Pi_{i=1}^{j-1}\varphi_{k_i}'(g)| \leq \delta_0^2/2^{j+1}, \forall g \in K_j$. Thus $\{\Pi_{i=1}^j \varphi_{k_i}'(g)\}_j$ is Cauchy, $\forall g \in G$. For each $1 \leq n \leq m$, denote $\varphi_n^m = \Pi_{i=n}^m \varphi_{k_i}'$, $\varphi_n^\infty = \Pi_{i=n}^\infty \varphi_{k_i}'$, $\varphi = \Pi_{i=1}^\infty \varphi_{k_i}'$. Then φ is a positive definite function on G such that $\varphi \geq 0$, $\varphi(1) = 1$, and

(b) $$|\varphi(g) - 1| \leq \Sigma_{m=1}^\infty |\varphi_1^m - \varphi_1^{m-1}| \leq \delta_0^2 \Sigma_{m=1}^\infty 2^{-m-1} = \delta_0^2/2,$$

$\forall g \in F_0$. Hence if we denote $(\pi_\varphi : G \to \mathcal{U}(\mathcal{H}_\varphi), \xi_\varphi)$ the pointed unitary representation associated with φ, then $\|\pi_\varphi(g)\xi_\varphi - \xi_\varphi\|^2 = 2 - 2\varphi(g) \leq \delta_0^2$ for all $g \in F_0$. Thus (A.3) applies to get a unit vector $\xi_0 \in \mathcal{H}_\varphi$ such that $\pi_\varphi(h)\xi_0 = \xi_0, \forall h \in H$.

Since ξ_φ is a cyclic vector for $\mathcal{H}_\varphi$ there exists $Y \subset \mathbb{C} \times G$ finite such that $\xi = T_Y(\xi_\varphi)$ satisfies $\|\xi_0 - \xi\| \leq \varepsilon/3$ and $\|\xi\| = 1$. Then we have

$$|1 - T_Y(\varphi)(h)| = |1 - \langle \pi_\varphi(h)\xi, \xi\rangle| \leq$$

(c) $$|\langle \pi_\varphi(h)\xi_0, \xi_0\rangle - \langle \pi_\varphi(h)\xi_0, \xi\rangle| + |\langle \pi_\varphi(h)\xi_0, \xi\rangle - \langle \pi_\varphi(h)\xi, \xi\rangle| \leq$$

$$\|\xi_0 - \xi\| + \|\xi_0 - \xi\| \le \varepsilon,$$

for all $h \in H$.

Using the inequality $|\varphi'(g_1) - \varphi'(g_2)|^2 \le 2\varphi'(1)(\varphi'(1) - Re\varphi'(g_1^{-1}g_2))$, for positive definite functions φ' on G and $g_1, g_2 \in G$, together with the fact that $\varphi_n^\infty \to 1$, it follows that $\|\varphi_n^\infty(g_1 \cdot g_2) - \varphi_n^\infty(\cdot)\|_\infty \to 0, \forall g_1, g_2 \in G$. Thus

$$\text{(d)} \qquad \lim_{n\to\infty} \|T_Y(\varphi) - \varphi_{n+1}^\infty T_Y(\varphi_1^n)\|_\infty = 0$$

Also since $\varphi_n^\infty(b'_{k_m}) \le \varphi'_{k_m}(b'_{k_m}), \forall m \ge n$ it follows that

$$\text{(e)} \qquad \lim_{m\to\infty} |(\varphi_{n+1}^\infty T_Y(\varphi_1^n))(b'_{k_m})| = \lim_{m\to\infty} |\varphi_{n+1}^\infty(b'_{k_m}) T_Y(\varphi_1^n)(b'_{k_m})| = 0,$$

for all $n \ge 1$. Combining (d) and (e) we have

$$\text{(f)} \qquad \lim_{m\to\infty} |T_Y(\varphi)(b'_{k_m})| = 0$$

But this contradicts (c) for $\varepsilon < 1$.

□

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